Ocean 620
Internal Gravity Waves

Motivation

If you randomly pick a location in the ocean, moor a current meter there for a few days, and look at the record, chances are that most of the variance will be due to motions in the broad category of internal gravity waves. If you focus on the variance at frequencies higher than about a cycle per day, then it is almost certain that the bulk of this variance will be due to internal waves.

Internal waves can exist in any stratified fluid. They are simply perturbations of the density field (or interface height, in a layered medium) that propagate in a wave-like manner. For the internal waves that we will consider first, the restoring forces are gravity and Coriolis. These waves are internal gravity waves; in the limit of nearly horizontal motion, so that Coriolis is much more important than gravity as a restoring force, they are called inertial or near-inertial waves. Other types of internal waves, to be considered in later lectures, are internal Kelvin waves (gravity waves strongly influenced by rotation and constrained to propagate along coastlines or the equator) and Rossby waves (which can be thought of as potential vorticity waves).

The way we describe the vertical structure of internal waves depends on their vertical scale. If the scale is very small compared to the water depth, then it is usually most efficient to think of the waves as propagating vertically and horizontally in a stratified fluid of infinite extent. If the vertical scale of the waves is not small compared to the water depth, it is often useful to describe them as horizontally propagating waves with a standing wave structure in the vertical—vertical normal modes. In the present lecture we will concentrate on the vertically propagating description.

The importance of internal gravity waves is two-fold:

1. They are a large part of the "noise" in the ocean that clutters up measurements when one is trying to discern the large-scale, low frequency circulation.

2. They play a major role in ocean mixing, that is, in the complex process that we try to sweep under the rug as "eddy diffusivity".

The role of internal waves as noise is fairly well understood, but is sometimes forgotten by people planning or studying measurements. The role of internal waves in mixing is still not very well understood, and is among the most active research areas in PO.

Why do internal waves exist?

Imagine a fluid parcel (or a water balloon) in a stratified fluid. If that parcel is displaced upward from its equilibrium position it will be less buoyant than the surrounding fluid, so it will sink when released. When it gets back to its equilibrium position it will still have downward momentum, so it will overshoot. Hence it will oscillate. The frequency of oscillation is called the *Brunt-Vaisala* or *buoyancy* frequency, and is always denoted by the letter N. If the fluid were perfectly incompressible, the frequency would be given by

$$N^2 = -\frac{g}{\rho}\frac{\partial \rho}{\partial z}. \tag{1}$$

This should look very reasonable to you: the frequency of an oscillator is proportional to the square root of the restoring force per unit mass, which in this case is proportional to the density gradient and to g. Taking into account compressibility, it turns out that

$$\begin{aligned} N^2 &= -\frac{g}{\rho}\left[\frac{\partial \rho}{\partial z} - \left(\frac{\partial \rho}{\partial z}\right)_{\text{adiabatic}}\right] && (2)\\ &= -\frac{g}{\rho}\left[\frac{\partial \rho}{\partial z} + \frac{g\rho}{C^2}\right] && (3)\end{aligned}$$

where C is the speed of sound. The higher the speed of sound, the less compressible the fluid must be, and the smaller the difference between the *in situ* density gradient and the adiabatic density gradient.

A better way to remove the effect of compressibility is to estimate the derivative at each depth by differencing the potential density of parcels slightly above and below that depth, where "potential" means that density is calculated *after* moving each parcel adiabatically to the depth in question. Numerically, this allows much more accurate estimates when N is small; and conceptually, it is a clearer statement of what what controls the buoyancy frequency.

Values of N in the ocean range from one cycle per 5 minutes or less in the equatorial thermocline to roughly 1 cycle per 3 hours at 5000 m. Of course, N is zero in homogeneous areas such as the surface mixed layer (when it is indeed thoroughly mixed).

What do internal waves look like, and how do they propagate?

Fluid displacements do not have to be vertical; one can imagine planes of fluid tilted at some angle θ from the vertical. (Note that this *theta* is then the magnitude of the elevation angle of the wavenumber from the horizontal.) This reduces the restoring force by $\cos^2\theta$: one factor of $\cos\theta$ comes from taking the component of gravity in the direction of motion, the other from taking that component of the density gradient. The frequency of oscillation, ω, then goes as $\cos\theta$. Neglecting the rotation of the earth, we have

$$\omega = N\cos\theta. \tag{4}$$

As the planes of motion become more horizontal ($\theta \rightarrow \pi/2$), however, the Coriolis force comes into play. When the planes are perfectly horizontal we have **inertial oscillations**, in which particles move in anticyclonic circular trajectories with $\omega = f$. Generalizing, for any value of $0 < \theta < \pi/2$, the particle trajectory in the tilted plane will be an ellipse with frequency $N > \omega > f$. The ellipse becomes a vertical line at $\omega = N$, $\theta = 0$, and broadens out into a circle at $\omega = f$, $\theta = \pi/2$. Recall that f is one cycle per 24 hours at 30° latitude, and about one cycle per 33 hours near Honolulu. Quantitatively, taking f into account, we have

$$\omega^2 = (N\cos\theta)^2 + (f\sin\theta)^2. \tag{5}$$

Just as with surface waves, we are considering a mathematical idealization. For surface waves this was a wave that was sinusoidal in x (say) and constant in y, extending to infinity in both directions. For internal waves we are imagining the fluid is unbounded in all directions. Now the disturbance is sinusoidal in x and z, and uniform in y (because we align our coordinate system so that the horizontal component of propagation is in the x direction). For simplicity we are also requiring that N be a constant. The velocity field and displacement, η, can then be expressed as

$$u = u_0\cos(kx + mz - \omega t) \tag{6}$$

$$v = u_0\frac{f}{\omega}\sin(kx + mz - \omega t) \tag{7}$$

$$w = -u_0\frac{k}{m}\cos(kx + mz - \omega t) \tag{8}$$

$$\eta = -u_0\,\omega\frac{k}{m}\sin(kx + mz - \omega t) \tag{9}$$

with the dispersion relation

$$\frac{m^2}{k^2} = \frac{N^2 - \omega^2}{\omega^2 - f^2}. \tag{10}$$

Note that the waves now propagate in the horizontal and the vertical; the vertical component of the wavenumber vector is m, the horizontal component is k.

From the dispersion relation we can, of course, calculate the phase velocity and the group velocity. The phase velocity vector is aligned with the wavenumber vector, normal to the planes of motion, that is, normal to the velocity vector at any time. The group velocity vector, on the other hand, lies on the intersection of the plane of motion and a normal vertical plane; that is, it is either up or down the plane of motion.

Now, here is a peculiar feature of all internal waves: *the vertical component of the group velocity is always opposite to the vertical component of the phase velocity.* Suppose you have an array of current meters in the vertical, so that you can measure the vertical component of the phase velocity. If it is *upward*, so that an extremum in u, say, occurs at successively later times at shallower current meters, then the internal wave *energy* must be propagating *downward*.

Another peculiar feature of internal waves is the way they reflect off a flat surface. Frequency is conserved in a reflection, as is the component of the wavenumber vector *along* the surface. For a given frequency, however, the angle between the wavenumber vector and the vertical is also fixed. The result is that the angle of incidence *does not* in general equal the angle of reflection. Wave energy can be trapped in a corner, leading to large amplitudes and nonlinear, dissipative behavior (breaking). It can also be trapped along a slope, leading to increased energy and dissipation there. This may be one of the main mechanisms by which mixing occurs in the ocean.

For small-scale internal waves, the variation of N with depth can be handled with a common mathematical technique called the WKB method. N is considered as varying slowly compared to the vertical scale of the wave ($1/m$), so rays of wave energy are simply refracted. This is analogous to the refraction of surface waves by a gradually shoalling ocean bottom.

Where do internal waves come from?

No one knows completely; there is a roughly isotropic, homogeneous field of internal wave noise in the ocean, with waves going every which way, so it is not at all clear where the sources are. Here are some of the processes involved, however:

Tides moving over topography generate internal tides, that is, internal waves at tidal frequencies, and at higher harmonics via nonlinear processes.

Surface waves can interact with each other to generate internal waves.

Wind changes excite near-inertial waves. As these propagate to lower latitudes the local inertial frequency decreases while the frequency of the waves remains unchanged, so they are no longer "near-inertial".

Convection in the mixed layer sends blobs of fluid bouncing off the base of the mixed layer, exciting internal waves that propagate away both downward and horizontally.

Instabilities due to fluid shear also cause turbulence, some of which radiates internal wave energy away from the disturbance.

Nonlinear interaction of existing internal waves causes exchanges of energy that tend to even out the spectrum.

The typical frequency-wavenumber spectrum of internal wave energy—the ocean noise field—has been described and approximated as a simple analytical form by Chris Garrett and Walter Munk. Their model is called the GM spectrum. This model is purely empirical—it is a simplified mathematical description of the observed internal wave spectrum. It does not address the question of *why* such a spectrum exists so widely in the ocean.

Where do internal waves go?

Internal wave energy must all be dissipated somewhere, and it is not clear how much dissipation occurs where. One dissipation mechanism is wave breaking, analogous to the breaking of surface waves. The superposition of various internal waves, with perhaps a contribution from the larger scale current field, can lead to an instability and thence to turbulent mixing. As already noted, this process may be greatly enhanced along ocean boundaries.

Internal waves do not always act to dissipate mean flows, in the normal sense of an eddy diffusivity; they can actually accelerate larger scale flows. Places where this occurs are called *critical layers*. They involve refraction of internal waves by the shear of a larger scale flow, to the point where the waves can no longer exist. The momentum they carry is then dumped into the mean flow.

Math: phase and group velocity

The dispersion relation for internal waves in an infinite domain with uniform stratification (10) can be rearranged as

$$\omega^2 = \frac{k^2 N^2 + m^2 f^2}{k^2 + m^2}. \tag{11}$$

Taking the square root of both sides would put it in the conventional form, specifying frequency as a function of the wavenumber components, but leaving it squared simplifies the calculation of group velocity. Taking the partial derivative of (11) with respect to k, we have

$$\begin{aligned} 2\omega \frac{\partial \omega}{\partial k} &= \frac{2kN^2(k^2+m^2) - 2k(k^2N^2 + m^2 f^2)}{(k^2+m^2)^2} \\ &= \frac{2km^2(N^2 - f^2)}{(k^2+m^2)^2}. \end{aligned} \tag{12}$$

For the vertical wavenumber component, skipping the first step, we have

$$2\omega \frac{\partial \omega}{\partial m} = \frac{-2mk^2(N^2 - f^2)}{(k^2+m^2)^2}. \tag{13}$$

Therefore the group velocity vector is

$$\vec{c_g} = (km^2, -mk^2)\left[\frac{N^2 - f^2}{\omega(k^2+m^2)}\right]. \tag{14}$$

The phase velocity vector is

$$\vec{c_p} = (k, m) \left[\frac{1}{\omega} \right]. \tag{15}$$

Now it is easy to see that $\vec{c_p} \cdot \vec{c_g} = 0$, so the group velocity is perpendicular to the phase velocity. Furthermore, the expressions in square brackets are positive (we define ω as positive so that all directional information is in the signed wavenumber components), so the horizontal components of $\vec{c_p}$ and $\vec{c_g}$ have the same sign, but the vertical components have opposite sign.